1 a)

$$N = 2 \sum_{|\vec{k}|<k_F} 1 = 2\left(\frac{L}{2\pi}\right)^2 \int_{|\vec{k}|<k_F} d^2k$$

$$= \frac{L^2}{\pi}\int_0^{k_F} dk\, k = \frac{L^2}{2\pi} k_F^2$$

$$k_F^2 = 2\pi \frac{N}{L^2} \qquad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} 2\pi \frac{N}{L^2}$$

b)

$$N = 2\sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1} = \frac{L^2}{\pi}\int_0^\infty dk\, k \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1}$$

$$d\varepsilon = \frac{\hbar^2}{m} k\, dk$$

$$N = \frac{L^2}{\pi}\frac{m}{\hbar^2}\int_0^\infty d\varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \frac{L^2}{\pi}\frac{m}{\hbar^2} k_B T \times \ln\left(1+e^{\beta\mu}\right)$$

$$\Rightarrow 1+e^{\beta\mu} = \exp\left[\frac{N}{L^2}\frac{\hbar^2}{m}\frac{\pi}{k_B T}\right]$$

$$\beta\mu = \ln\left\{\exp\left[\frac{N}{L^2}\frac{\hbar^2}{m}\frac{\pi}{k_B T}\right]-1\right\}$$

c) $T \to 0$: $\exp[\ ]$ gets large $\Rightarrow$ neglect 1

$$\beta\mu = \frac{N}{L^2}\frac{\hbar^2}{m}\frac{\pi}{k_B T}$$

$$\mu = \frac{N}{L^2}\frac{\hbar^2}{m}\pi = \varepsilon_F \quad \checkmark$$

2) a) $$Z_1 = \int d\varphi \sin\theta \, d\theta \quad e^{+\beta M g \ell \cos\theta}$$

$$= 2\pi \int_{-1}^{1} d\eta \; e^{-\beta M g \ell \eta} = \frac{2\pi}{\beta M g \ell}\left(e^{\beta M g \ell} - e^{-\beta M g \ell}\right)$$

$$Z_1 = \frac{4\pi}{\beta M g \ell} \sinh(\beta M g \ell)$$

b) $$Z_N = Z_1^N$$

$$F = -k_B T \ln Z_N = -N k_B T \ln Z_1$$

$$= -N k_B T \ln\left[\frac{4\pi}{\beta M g \ell} \sinh(\beta M g \ell)\right]$$

c) $$L_z = N\ell \langle \cos\theta \rangle = \left(+\frac{\partial}{\partial(\beta M g \ell)} \ln Z_1\right) N\ell$$

$$= N\ell \frac{+\partial}{\partial(\beta M g \ell)}\left[-\ln(\beta M g \ell) + \ln\sinh(\beta M g \ell)\right]$$

$$= N\ell \left[\frac{-1}{\beta M g \ell} + \coth(\beta M g \ell)\right]$$

d) $T \to 0,\ \beta \to \infty$: $\quad L_z = N\ell\,[0 + 1] = N\ell$

$T \to \infty,\ \beta \to 0$: $\quad L_z = N\ell\,(\beta M g \ell)/3$

$$\coth(x) = \frac{1}{x} + \frac{x}{3} \cdots$$

10:02 e) $L_z = \frac{1}{3} N \ell^2 \beta \quad Mg$

(3) a)

$$U = 2\sum_{\vec{k}} \varepsilon_{\vec{k}} n_{\vec{k}} = 2\left(\frac{L}{2\pi}\right)^2 \int d^2k \;\; \varepsilon_{\vec{k}} n_{\vec{k}}$$

$$= \frac{L^2}{\pi} \int dk\, k \;\; \varepsilon_{\vec{k}} n_{\vec{k}} \qquad\qquad d\varepsilon = \hbar c\, dk$$

$$= \frac{L^2}{\pi} \frac{1}{\hbar^2 c^2} \int d\varepsilon\, \varepsilon \;\; \varepsilon\, n(\varepsilon)$$

$$g(\varepsilon) = \frac{L^2}{\pi} \frac{1}{\hbar^2 c^2} \varepsilon$$

b)

$$U = \frac{L^2}{\pi} \frac{1}{\hbar^2 c^2} \int_0^\infty d\varepsilon \; \frac{\varepsilon^2}{e^{\beta\varepsilon} - 1}$$

$$= \frac{L^2}{\pi} \frac{(k_B T)^3}{\hbar^2 c^2} \quad 2\,\zeta(3)$$

c)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3 \frac{L^2}{\pi} \frac{k_B^3 T^2}{\hbar^2 c^2} \; 2\zeta(3) \quad \sim T^2$$

(4) a) $\mu B \gg J$ spins $\parallel$ field

$J \gg \mu B$ FM

b) $$H = -\sum_i S_i (6Jm + \mu B)$$

c) $$Z_N = Z_1^N \qquad Z_1 = \sum_{S=-1,0,1} e^{\beta(6Jm+\mu B)S_i}$$

$$= 1 + 2\cosh\beta(6Jm + \mu B)$$

$$m = \frac{\partial}{\partial(\beta\mu B)} \ln Z_1 = \frac{2\sinh\beta(6Jm+\mu B)}{1+2\cosh\beta(6Jm+\mu B)}$$

d) $B = 0$, expand for small m

$$m = \frac{12\beta J m}{3} \qquad k_B T_c = 4J$$