Physics 5413: Chaos and Dynamics – Project 4a

due date: Friday, March 11, 2022

Predator - Prey Model of Odell (100 points + 10 bonus points)
(Problem 8.2.8 from Strogatz' book)

Consider the following model for the population dynamics of two species, a predator and a prey species (Odell 1980):

$$\begin{aligned} \dot{x} &= x\left[x(1-x)-y\right] \\ \dot{y} &= y(x-a) \end{aligned}$$

where $x \geq 0$ is the dimensionless population of the prey and $y \geq 0$ is the dimensionless population of the predator. $a \geq 0$ is a control parameter.

1. What is the biological meaning of the parameter a?
2. Find the fixed points, study their stability and discuss their biological meaning.
3. Sketch the state space trajectories for $a > 1$, and show that the predators go extinct.
4. Show that a Hopf bifurcation exists at $a_c = 1/2$. Is it subcritical or supercritical?
5. Estimate the frequency of the limit cycle oscillations for a near the bifurcation.
6. Sketch state space trajectories for all qualitatively distinct cases for $0 < a < 1$.

BONUS problem: Infinite-period bifurcation (10 BONUS points)

The Hopf bifurcation is not the only way to destroy a limit cycle. In this problem you will explore a different type of limit-cycle bifurcation not covered in class. Consider the following system:

$$\begin{aligned} \dot{x_1} &= -\mu x_2 + x_1(1-x_1^2-x_2^2) + \frac{x_2^2}{\sqrt{x_1^2+x_2^2}} \\ \dot{x_2} &= \mu x_1 + x_2(1-x_1^2-x_2^2) - \frac{x_1 x_2}{\sqrt{x_1^2+x_2^2}} \end{aligned}$$

where μ is the control parameter.

1. Identify fixed points and limit cycles. Use whatever method comes handy: numerical integration of the equations of motion, constructing a trapping region, or a direct analytical analysis. You may want to transform to polar coordinates (r, Θ) with $x_1 = r\cos\Theta,\ x_2 = r\sin\Theta$.
2. Show that a bifurcation exists at $\mu = 1$. Describe the behavior above and below the bifurcation. Sketch the state space trajectories for both cases.
3. Why is this bifurcation called an infinite-period bifurcation?