Physics 5403: Computational Physics – Project 3b

due date: Tuesday, Sep 27, 2022

Chaos in the driven damped pendulum

Consider the motion of a driven damped pendulum governed by the differential equation

$$\frac{d^2\Theta}{dt^2} + q\frac{d\Theta}{dt} + \sin(\Theta) = b\ \cos(\omega_0 t)\ .$$

Simulate the motion of this pendulum for different parameter sets and discuss the resulting behavior.

a) Write a program which integrates the above ordinary differential equation from time t_{min} to time t_{max} using a time step τ. Record the angle Θ, the angular velocity and the total energy as functions of time.

b) First study the motion for $q = b = 0$ (no damping, no driving force). Determine a reasonable value for the time step τ. What would be a good initial guess? Why? Optimize τ using the requirement of energy conservation. To this end, calculate the energy change over one period of the oscillation and plot it as a function of τ. What type of behavior do you expect? Choose an appropriate way of plotting the result.

c) Set $q = 1/2$ and $\omega_0 = 2/3$. Simulate the trajectory for different values of b ($b = 0,\ 0.5,\ 0.9,\ 1.2$). Plot angle and angular velocity as functions of time. Follow the trajectories for about 100 periods of the driving force. Plot the angular velocity vs. the angle (phase space plot). Describe what you observe.

d) For the above parameter sets calculate the Lyapunov exponent by observing two trajectories that start from very close initial conditions. The Lyapunov exponent λ is obtained by fitting the difference between the trajectories, $\log(\Delta\Theta)$ to a linear function $a + \lambda t$ of time. To get reliable results you should average $\log(\Delta\Theta)$ over several runs with different initial conditions before fitting.

e) A route to chaos: Repeated bifurcations:
Study trajectories for $q = 1/2$ and $\omega_0 = 2/3$ and values of b between 1.35 and 1.5 (for about 100 periods of the driving force). Observe the angle at discrete times in phase with the driving force (i.e. at times t given by $\omega_0 t = 2\pi n$ with n integer). What do you observe after the initial transients have decayed? Plot the resulting angle values vs. b. This is a bifurcation diagram.