Physics 481: Condensed Matter Physics - Homework Solutions 5

Problem 5.1

A B
A B A
A B A A B
A B A A B A B A

n_A	n_B	n_A/n_B
1	1	1
2	1	2
3	2	3/2
5	3	5/3

inflation $A \to AB$, $B \to A$

$$n_A^{(k+1)} = n_A^{(k)} + n_B^{(k)} \quad , \quad n_B^{(k+1)} = n_A^{(k)}$$

$$\frac{n_A^{(k+1)}}{n_B^{(k+1)}} = 1 + \frac{n_B^{(k)}}{n_A^{(k)}}$$

limit $k \to \infty$: $\dfrac{n_A^{(k)}}{n_B^{(k)}} \to x$

$$x = 1 + \frac{1}{x} \qquad x^2 - x - 1 = 0 \qquad x = \frac{1}{2} \pm \sqrt{5/4}$$

$$\boxed{x = \frac{1}{2}\left(1 + \sqrt{5}\right)} \qquad \text{golden mean}$$

Problem 5.2

a) $$E = 2N \sum_{j=1}^{\infty} (-1)^j \frac{q^2}{jR} + 2N \frac{A}{R^n}$$

$$= 2N \left\{ -\frac{q^2}{R} \ln 2 + \frac{A}{R^n} \right\}$$

$$\frac{\partial E}{\partial R} = 2N \left\{ \frac{q^2}{R^2} \ln 2 - n \frac{A}{R^{n+1}} \right\} \stackrel{!}{=} 0$$

$$R_0^{n-1} = \frac{nA}{q^2 \ln 2} \qquad \boxed{R_0 = \left(\frac{nA}{q^2 \ln 2} \right)^{1/n-1}}$$

b) $$E_0 = 2N \left\{ -q^2 \ln 2 \left(\frac{q^2 \ln 2}{nA} \right)^{\frac{1}{n-1}} + A \left(\frac{q^2 \ln 2}{nA} \right)^{\frac{n}{n-1}} \right\}$$

$$= -2N \ln 2 \; q^2 \left\{ \frac{1}{R_0} - \frac{1}{n} \frac{1}{R_0} \right\}$$

$$\boxed{E_0 = -2N \ln 2 \; q^2 \frac{1}{R_0} \left(1 - \frac{1}{n} \right)}$$

c) $$E = E(R_0) + \frac{1}{2} (\delta R_0)^2 \left(\frac{\partial^2 E}{\partial R^2} \right)_{R_0}$$

$$\frac{\partial^2 E}{\partial R^2} = 2N \left\{ -\frac{2q^2}{R^3} \ln 2 + \frac{n(n+1) A}{R^{n+2}} \right\}$$

$$\left. \frac{\partial^2 E}{\partial R^2} \right|_{R_0} = 2N \left\{ -\frac{2q^2 \ln 2}{R_0^3} + \frac{(n+1)\, q^2 \ln 2}{R_0^3} \right\}$$

$$= \frac{2N \ln 2 \; q^2}{R_0^3} (n-1)$$

$$\boxed{\Delta E = \frac{1}{2}\delta^2 \quad \frac{2N \ln 2\, z^2}{R_0}\,(n-1)}$$

Problem 4.3 (1)

a) $$p \sim e^{-\sum_{i=1}^{N} K\theta_i^2/k_BT}$$

normalization $$\int_{-\infty}^{\infty} e^{-K\theta^2/k_BT} = \sqrt{2\pi \frac{k_BT}{2K}} \qquad (K \gg k_BT)$$

$$p = \left(\frac{K}{\pi k_BT}\right)^{N/2} e^{-\sum_{i=1}^{N} K\theta_i^2/k_BT}$$

b) $$X_N = a\left[\cos\theta_1 + \cos(\theta_1+\theta_2) + \dots + \cos(\theta_1+\dots+\theta_N)\right]$$

$$Y_N = a\left[\sin\theta_1 + \dots + \sin(\theta_1+\dots\theta_N)\right]$$

c) $$X_N^2 = a^2 \sum_{jk} \cos\left(\sum_{l=1}^{j}\theta_l\right) \cos\left(\sum_{l'=1}^{k}\theta_{l'}\right)$$

$$= \frac{a^2}{4}\sum_{jk}\left(e^{i\sum_{l=1}^{j}\theta_l + i\sum_{l'=1}^{k}\theta_{l'}} + e^{i\sum_{l=1}^{j}\theta_l - i\sum_{l'=1}^{k}\theta_{l'}} + c.c.\right)$$

$$\sqrt{\frac{K}{\pi k_BT}}\int d\theta\, e^{-K\theta^2/k_BT + i\theta} = e^{-\frac{1}{4}\frac{k_BT}{K}}$$

$$\sqrt{\frac{K}{\pi k_BT}}\int d\theta\, e^{-K\theta^2/k_BT + 2i\theta} = e^{-\frac{k_BT}{K}}$$

(2)

$$\left\langle e^{i\sum_{l=1}^{j}\theta_l + i\sum_{l'=1}^{k}\theta_{l'}}\right\rangle = e^{-j\frac{k_BT}{K}}\, e^{-(k-j)\frac{k_BT}{4K}} \qquad (j<k)$$

$$= e^{-k\frac{k_BT}{K}}\, e^{-(j-k)\frac{k_BT}{4K}} \qquad (k<j)$$

$$\left\langle e^{-i\sum_{l}^{j}\theta_l - i\sum_{l'}^{k}\theta_{l'}}\right\rangle = e^{-|j-k|\frac{k_BT}{4K}}$$

$$\langle x_N^2\rangle = a^2\frac{1}{2}\sum_{j<k}\left(2e^{-j\frac{k_BT}{K}-(k-j)\frac{k_BT}{4K}} + 2\,e^{-(k-j)\frac{k_BT}{4K}}\right)$$

$$+\, a^2\frac{1}{4}\sum_j\left(2e^{-j\frac{k_BT}{\theta}} + 2\right)$$

Carry out sum over $k-j$, extend to infinity

$$\sum_{x=1}^{\infty} e^{-x\frac{k_BT}{4K}} = \frac{1}{1-e^{-\frac{k_BT}{4K}}} - 1 = \frac{e^{-\frac{k_BT}{4K}}}{1-e^{-\frac{k_BT}{4K}}}$$

$$\langle x_N^2\rangle = a^2\sum_j\left(e^{-j\frac{k_BT}{K}} + 1\right)\frac{e^{-k_BT/4K}}{1-e^{-k_BT/4K}}$$

$$+\frac{a^2}{2}\sum_j\left(e^{-j\frac{k_BT}{K}} + 1\right)$$ Neglect exponential in bracket

$$\langle x_N^2\rangle = Na^2\left(\frac{1}{2} + \frac{e^{-k_BT/4K}}{1-e^{-k_BT/4K}}\right)$$

$$\langle x_N^2\rangle = \frac{N}{2}a^2\left(\frac{1+e^{-k_BT/4K}}{1-e^{-k_BT/4K}}\right)$$

$$\boxed{\tilde{a}^2 = \frac{1}{2}\,\frac{1+e^{-k_BT/4K}}{1-e^{-k_BT/4K}}}$$