7.1

$$S = -k_B \sum_i \sum_j p(i,j) \ln p(i,j)$$

statistical independence $p(i,j) = p_1(i)\, p_2(j)$

$$S = -k_B \sum_i \sum_j p_1(i)\, p_2(j) \left[\ln p_1(i) + \ln p_2(j) \right]$$

$$= -k_B \sum_i p_1(i) \ln p_1(i) \sum_j p_2(j)$$

$$-k_B \sum_i p_1(i) \sum_j p_2(j) \ln p_2(j)$$

$$= -k_B \sum_i p_1(i) \ln p_1(i) - k_B \sum_j p_2(j) \ln p_2(j)$$

(using $\sum_i p_1(i) = \sum_j p_2(j) = 1$)

$$S = S_1 + S_2$$

7.2.

a) $$F = -k_B \sum_i p_i \ln_i - \lambda\left(\sum_i p_i - 1\right)$$

$$\frac{\partial F}{\partial p_j} = -k_B \sum_i \left(\delta_{ij} \ln p_i + p_i \frac{1}{p_i} \delta_{ij}\right) - \lambda \sum_i \delta_{ij}$$

$$= -k_B \ln p_j - k_B - \lambda \stackrel{!}{=} 0$$

$$\ln p_j = (-k_B - \lambda)/k_B \qquad p_j = e^{-\left(1 + \frac{\lambda}{k_B}\right)}$$

independent of j

$$\sum_{j=1}^{N} p_j = 1 \quad \Rightarrow \quad p_j = e^{-\left(1 + \frac{\lambda}{k_B}\right)} = \frac{1}{N}$$

b) $$F = -k_B \sum_i p_i \ln p_i - \lambda\left(\sum_i p_i - 1\right) - \mu\left(\sum_i p_i E_i - \langle E\rangle\right)$$

$$\frac{\partial F}{\partial p_j} = -k_B \ln p_j - k_B - \lambda - \mu E_j$$

$$p_j = e^{-\left(1 + \frac{\lambda}{k_B} + \frac{\mu E_j}{k_B}\right)}$$

structure of Boltzmann distribution

7.3 a) • heat leaking into the house $\Delta Q = A(T_h - T_e)$

• heat removed from inside (low-T reservoir) : Q_e

Carnot cycle running backwards

$Q_e > 0\ ,\ Q_h < 0 \qquad W + Q_e + Q_h = 0$

$$\frac{Q_e}{|Q_h|} = \frac{T_e}{T_h} \qquad Q_h = -Q_e \frac{T_h}{T_e}$$

$$W + Q_e - Q_e \frac{T_h}{T_e} = W + Q_e \frac{T_e - T_h}{T_e} = 0$$

$$\frac{Q_e}{W} = \frac{T_e}{T_h - T_e} \qquad Q_e = \frac{T_e}{T_h - T_e} W$$

• in steady state: $\Delta Q = Q_e$

$$A(T_h - T_e) = \frac{T_e}{T_h - T_e} E \qquad (W = E)$$

$$A(T_h - T_e)^2 = T_e E$$

$$A T_e^2 - (2AT_h + E) T_e + A T_h^2$$

$$T_e^2 - \left(2T_h + \frac{E}{A}\right) T_e + T_h^2 = 0$$

$$T_e = T_h + \frac{E}{2A} \pm \sqrt{\left(T_h + \frac{E}{2A}\right)^2 - T_h^2}$$

↑ pick − $(T_e < T_h)$

7.3 b)

$$A\left(T_h - T_e\right)^2 = T_e E$$

$T_e = 72°F$

$T_h = 85°F$

$$A\left(T_h' - T_e\right)^2 = T_e 2E$$

$$\left(T_h - T_e\right)^2 = \left(T_h' - T_e\right)^2 \frac{1}{2}$$

$$T_h' - T_e = \sqrt{2}\left(T_h - T_e\right) = 18.4°F$$

$$T_h' = 90.4°F$$