12.1

Contribution of d.o.f. to partition function

$$Z = \int_{-\infty}^{\infty} dq \; e^{-\beta \frac{A}{2} |q|^n}$$

$$= 2\int_0^{\infty} dq \; e^{-\beta \frac{A}{2} q^n}$$

Gamma function

use $\int_0^{\infty} dx \; e^{-ax^n} = \frac{1}{n}\Gamma\left(\frac{1}{n}\right) a^{-\frac{1}{n}}$

$$Z = 2 \frac{1}{n} \Gamma\left(\frac{1}{n}\right) \left(\frac{\beta A}{2}\right)^{-\frac{1}{n}}$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = -\frac{\partial}{\partial \beta} \ln\left(\beta^{-\frac{1}{n}}\right)$$

$$\langle E \rangle = \frac{1}{n}\frac{1}{\beta} = \frac{1}{n} k_B T$$

12.2 a) in real space, use cylindrical coordinates

$r, \theta, z \qquad r = \sqrt{x^2+y^2} \qquad H_1 = \frac{\vec{p}^2}{2m} - \frac{m}{2}\omega^2 r^2$

$$Z_N = \frac{1}{N!} z_1^N \qquad z_1 = \int \frac{d^3p\, d^3r}{h^3} e^{-\beta \frac{\vec{p}^2}{2m} + \frac{\beta}{2} m\omega^2 r^2}$$

$$= z_{kin}\, z_{pot}$$

$$z_{kin} = \frac{1}{h^3} \sqrt{2\pi m k_B T}^3 \quad \text{as in class}$$

$$z_{pot} = \int_0^R r\,dr \int_0^{2\pi} d\theta \int_0^H dz\; e^{\beta m \omega^2 r^2/2} \qquad u = \frac{r^2}{2}$$

$$= 2\pi H \int_0^{R^2/2} du\; e^{\beta \omega^2 u m} = \frac{2\pi H}{m\beta\omega^2}\left[e^{\beta m\omega^2 R^2/2} - 1\right]$$

b) $$U = -\frac{\partial}{\partial\beta} \ln Z = -N \frac{\partial}{\partial\beta} \ln z_1 = -N \frac{\partial}{\partial\beta} \ln\left(z_{kin}\, z_{pot}\right)$$

$$= \underbrace{\frac{3}{2} N k_B T}_{\text{kinetic}} + k_B T - \frac{(m\omega^2 R^2/2)\, e^{\beta m\omega^2 R^2/2}}{e^{\beta m\omega^2 R^2/2} - 1}$$

c) $$n(r) \sim e^{\frac{\beta}{2} m\omega^2 r^2} \qquad \text{normalize}$$

$$n(r) = \frac{m\beta\omega^2}{e^{\beta m\omega^2 R^2/2} - 1}\, e^{\beta/2\, m\omega^2 r^2}$$

12.3. a) $Z_N = \frac{1}{N!} Z_1^N$

$$Z_1 = \int \frac{d^3p\, d^3q}{h^3} e^{-\beta c|\vec{p}|} = \frac{V}{h^3} \int d^3p\, e^{-\beta c|\vec{p}|}$$

use spherical coordinates

$$Z_1 = \frac{V}{h^3} 4\pi \int_0^\infty dp\, p^2 e^{-\beta c p} = \frac{4\pi V}{h^3} \frac{\partial^2}{\partial(\beta c)^2} \int_0^\infty dp\, e^{-\beta p c}$$

$$= \frac{4\pi V}{h^3} \frac{2}{(\beta c)^3}$$

$$F = -k_B T \ln Z_N = -N k_B T \left\{ \ln\left[\frac{4\pi V}{h^3 N} \frac{2}{(\beta c)^3} \right] + 1 \right\}$$

b) $p = -\left(\frac{\partial F}{\partial V}\right)_T = N k_B T \frac{1}{V} \Rightarrow pV = N k_B T$

as in nonrelativistic case

c) $U = -\frac{\partial}{\partial \beta} \ln Z = N 3 \frac{1}{\beta} = 3 N k_B T$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3 N k_B$$

d) $C_p = \left(\frac{\delta Q}{\partial T}\right)_p = \frac{dU + p\,dV}{dT} = 3 N k_B + N k_B$

$C_p = 4 N k_B \qquad \gamma = \frac{C_p}{C_V} = \frac{4}{3}$

(vs 5/3 for non-relativistic case)