11.1

a) $Z_1 = 1 + 2e^{-\beta\varepsilon}$

$Z_N = Z_1^N = \left(1+2e^{-\beta\varepsilon}\right)^N$

b) $\underline{U} = -\frac{d}{d\beta}\ln Z = -N\frac{d}{d\beta}\ln\left(1+2e^{-\beta\varepsilon}\right)$

$\underline{U} = N\frac{2\varepsilon e^{-\beta\varepsilon}}{1+2e^{-\beta\varepsilon}} = 2N\varepsilon\frac{1}{e^{\beta\varepsilon}+2}$

c) $p_{st} = \frac{1}{1+2e^{-\beta\varepsilon}}$

$\langle N_{st}\rangle = N p_{st} = \frac{N}{1+2e^{-\beta\varepsilon}}$

d) $T\to 0,\ \beta\to\infty$

$\langle N_{st}\rangle \to N$

$T\to\infty,\ \beta\to 0$

$\langle N_{st}\rangle \to N/3$

11.2 a) $Z_1 = 2 + e^{-\beta\varepsilon}$

$$P_1 = \frac{1}{2 + e^{-\beta\varepsilon}} = P_2$$

$$P_3 = \frac{e^{-\beta\varepsilon}}{2 + e^{-\beta\varepsilon}}$$

b) $Z_1 = 2 + e^{-\beta\varepsilon}$ $\qquad Z_N = (2 + e^{-\beta\varepsilon})^N$

(if distinguishable)

c) $F = -k_B T \ln Z = -k_B T N \ln(2 + e^{-\beta\varepsilon})$

$$U = -\frac{d}{d\beta} \ln Z = -N \frac{d}{d\beta} \ln(2 + e^{-\beta\varepsilon})$$

$$= N \frac{\varepsilon e^{-\beta\varepsilon}}{2 + e^{-\beta\varepsilon}} = \frac{N\varepsilon}{2e^{\beta\varepsilon} + 1}$$

$$C = \frac{dU}{dT} \quad \text{(no work)}$$

$$= \frac{d}{d\beta} U \left(\frac{d\beta}{dT}\right) = \frac{N\varepsilon^2 \, 2e^{+\beta\varepsilon}}{(2e^{\beta\varepsilon} + 1)^2} \frac{1}{k_B T^2}$$

$$= 2Nk_B \frac{(\beta\varepsilon)^2 e^{\beta\varepsilon}}{(2e^{\beta\varepsilon} + 1)^2}$$

d) $F = \underline{U} - TS$

$S = (\underline{U} - F)/T$

$$= \frac{1}{T}\frac{N\varepsilon}{2e^{\beta\varepsilon}+1} + k_B N \ln(2 + e^{-\beta\varepsilon})$$

$\beta \to \infty$, $T \to 0$: first term vanishes as $e^{\beta\varepsilon}$ diverges

$S \to N k_B \ln 2$ $\quad$ $\ln 2$ per atom due to doubly degenerate ground state